1) (10 points) For the following statements, answer by **TRUE** or **FALSE**. When **FALSE**, give a counterexample:

a. If $\sum a_n$ converges then it converges absolutely.

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False, ~~If~~ ~~If~~ If $\sum |a_n|$ converges then it converges absolutely.

counter exemple

$\sum (-1)^{n+1} \frac{1}{n}$ this series converges $U_n = \frac{1}{n} > 0$.

U_n is decreasing and $\lim_{n\to\infty} U_n = 0$

But

$\sum |(-1)^{n+1} \frac{1}{n}| = \sum \frac{1}{n}$ $p=1$

this series diverges

So $\sum (-1)^{n+1} \frac{1}{n}$ is conditionaly convergent

1

b. Given that $0 < f(x) < g(x)$ for all x, and $\int_1^{\infty} g(x)dx$ diverges, then $\int_1^{\infty} f(x)dx$ diverges.

1

false but we can say that if $\int_1^{\infty} f(x)dx$ diverges $\int_1^{\infty} g(x)$ diverges too

counter exemple

let $f(x) = \frac{1}{x}$ and $g(x) = x$

$\Rightarrow 0 < f(x) < g(x)$

$0 < \frac{1}{x} < x$

$\lim \int_1^{\infty} x\,dx = \infty \Rightarrow$ diverges

and but $\int \frac{1}{x}dx$ ~~converges~~ diverges

$$\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$$

$$\frac{3^{n+1} \, (n+1)!^2}{2(n+1)!} \times \frac{(2n)!}{3^n (n!)^2}$$

$$\frac{\cancel{3^{n+1}} \, (n+1)(n+1)\, \cancel{n!}\, \cancel{n!}}{\cancel{2(n+1)}\,\cancel{2n!} \quad (2n+2)(2n+1)\cancel{(2n)!}} \times \frac{\cancel{(2n)!}}{\cancel{3^n}\,\cancel{n!}\,\cancel{n!}}$$

$$3 \, \frac{n^2+\cancel{n}+\cancel{n}+1}{4n^2+4n+2\cancel{n}+2} = \frac{3}{4} < 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} + \frac{3}{2^n} \qquad \frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ldots \quad 0,248\ldots$$

$$+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}.$$

$$\cancel{\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{5^n} + \frac{3}{2^n}}$$

10

2) **(10 points)** Circle the letter of the correct answer.

a. Use the ratio test to determine if the following series converges or diverges. List the value of ρ:

$$\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$$

A) $\rho = \frac{8}{5}$; converges

B) $\rho = \frac{4}{5}$; converges

C) $\rho = \frac{3}{4}$; converges

D) $\rho = \frac{5}{4}$; diverges

5

b. The sum of the series $\sum_{n=1}^{\infty} \frac{1}{5^n} + \frac{3}{2^n}$ is:

A) $\frac{31}{20}$

B) $\frac{43}{60}$

C) $\frac{13}{4}$

D) $\frac{79}{100}$

$\ln(2n+1)'$

$\frac{u'}{u}$

$\frac{1}{2n+1} \times 2$

$$\left(\frac{n^2-4}{n^2}\right)^n$$

$$\left(1-\frac{4}{n^2}\right)$$

$$\left(1-\frac{2}{n}\right)^n \left(1+\frac{2}{n}\right)^n$$

$$e^{-2} \cdot e^2 = e^0 = 1$$

$$\ln (2n+1)^{1/n}$$

$$\frac{1}{n} \ln(2n+1) \quad \lim \frac{\ln(2n+1)}{n}$$

15

3\. (15 points) Determine whether the following sequence converges or diverges. If the sequence converges, find its limit.

a) (7 points) $a_n = \left(1 - \frac{4}{n^2}\right)^n$

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$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{4}{n^2}\right)^n$$

$$= \lim_{n\to\infty} \left(1 - \frac{2}{n}\right)^n \left(1 + \frac{2}{n}\right)^n$$

$$= e^{-2} \cdot e^{2} = e^{0}$$

$$= 1.$$

this sequence converges to 1.

b) (8 points) $a_n = (2n+1)^{1/n}$

8

$$\ln a_n = \frac{1}{n} \ln(2n+1)$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln(2n+1)}{n} = \frac{\infty}{\infty} \overset{\text{l'hop}}{=} \frac{\lim_{n\to\infty} \frac{2}{2n+1}}{1} = \lim_{n\to\infty} \frac{2}{2n+1}$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0.$$

$$\lim_{n\to\infty} \ln a_n = 0.$$

$$\lim_{n\to\infty} a_n = e^{0} = 1$$

this sequence also converges to 1

$$\int_1^\infty \frac{2+\cos x}{\cancel{x}}$$

$$\cancel{\int \frac{\cos x}{x^2}}$$

$$-1 < \cos x < 1$$

$$-\frac{1}{x}+2 < \frac{\cos x}{x} < \frac{1}{x}+2$$

$$1 < \cos x + 2 < 3$$

$$\frac{1}{\cancel{x}} < \frac{\cos x + 2}{x} < \frac{3}{\cancel{x}}$$

4) **(5 points)** Estimate the magnitude of the error involved in using the sum of the first five terms to approximate $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$. (Assume that the series satisfies the conditions of the Alternating Series Test).

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}$ let $U_n = (-1)^{n+1} \frac{1}{n!}$

$U_1 = (-1)^{1+1} \frac{1}{1!} = 1 \cdot 1 = 1.$ $U_3 = (-1)^{3+1} \frac{1}{3!} = 1 \cdot 6 = 6.$

$U_2 = (-1)^{2+1} \frac{1}{2!} = -1 \cdot 2 = -2.$ $U_4 = (-1)^{4+1} \frac{1}{4!} = -1 \cdot 24 = -24.$ $\frac{1}{\quad}$

$U_5 = (-1)^{5+1} \frac{1}{5!} = 1 \cdot 120 = 120.$ $U_6 = (-1)^{6+1} \frac{1}{6!} = \frac{1}{-1 \cdot 720} = \frac{1}{-720}$

$|\text{Error}| \leq \frac{1}{720}$

the error should be smaller than U_{n+1} (if we have to include the 1st 5 term the $U_{n+1} = U_6$) and it has the ~~opposite~~ same sign as u_{n+1} $\Rightarrow$ Error $< \frac{1}{720}$.

10

5) **(10 points)** Determine whether $\int_1^{\infty} \frac{2+\cos x}{x}\,dx$ converges or diverges. Give a reason for your answer.

$\lim_{a \to \infty} \int_1^a \frac{2+\cos x}{x}$

$-1 < \cos x < 1$

$1 < 2+\cos x < 3$

$\frac{1}{x} < \frac{2+\cos x}{x} < \frac{3}{x}.$

and $\int \frac{1}{x}\,dx$ is divergent then we can say that $\int_1^{\infty} \frac{2+\cos x}{x}\,dx$ diverges too.

$$\lim_{n\to\infty} \int_1^\infty \frac{n^{4/3}}{n^{5/3}}$$

$$\frac{n^{1/3+1}}{1/3+1} = \frac{3n^{4/3}}{4}$$

$$\lim_{n\to\infty} \frac{3n^{4/3}}{4} = \infty$$

$$\frac{1}{n} < \frac{1}{\ln n}.$$

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6) (20 points) Determine whether the following series converges absolutely, converges conditionally, or diverges. Justify your answer.

7

a) $\sum_{n=1}^{\infty} \frac{n^{4/3}\cos n\pi}{n^{5/3}}$

$\cos n\pi = (-1)^n$.

$\sum_{n=1}^{\infty} \frac{(-1)^n n^{4/3}}{n^{5/3}}$

this series converges because $U_n = \frac{n^{4/3}}{n^{5/3}} > 0$

U_n is decreasing

and $\lim_{n\to\infty} U_n = 0$

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n^{4/3}}{n^{5/3}} \right| = \sum_{n=1}^{\infty} \frac{n^{4/3}}{n^{5/3}}$

Let $f(x) = \frac{x^{4/3}}{x^{5/3}}$

$\lim_{\cancel{x}\to\infty} \int_1^{\infty} \frac{\cancel{x^{4/3}}}{\cancel{x^{5/3}}} = \lim_{x\to\infty} \frac{3x^{4/3}}{4} = \infty$ (integral test)

$\Rightarrow$ this series is conditionaly convergent

10

b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$

$\ln n < n$.

$\frac{1}{\ln n} > \frac{1}{n}$

$\sum \frac{1}{n}$ ~~is~~ ~~conver~~ diverges $(p=1)$ so $\sum \frac{1}{\ln n}$ diverges by the DCT

But

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is convergent because $U_n = \frac{1}{\ln n} > 0$.

U_n is decreasing

and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

so this series is conditionally convergent

$$\int_0^4 \frac{dx}{\sqrt{|x-3|}}$$

$$\int_0^3 \frac{dx}{\sqrt{|x-3|}} + \int_3^4 \frac{dx}{\sqrt{|x-3|}}$$

10

7) **(10 points)** Evaluate $\int_0^4 \frac{dx}{\sqrt{|x-3|}}$

$$= \int_0^3 \frac{dx}{\sqrt{|x-3|}} + \int_3^4 \frac{dx}{\sqrt{|x-3|}}$$

$$\lim_{b\to 3} \int_0^b \frac{dx}{\sqrt{|x-3|}} = \lim_{b\to 3} \int_0^b \frac{dx}{(-x+3)^{1/2}} = \lim_{b\to 3} \int_0^b (-x+3)^{-1/2}$$

$$= \lim_{b\to 3} -2 (-x+3)^{1/2} \Big]_0^b = \lim_{b\to 3} \left[-2(-b+3)^{1/2} + 2(-0+3)^{1/2}\right].$$

$$= \lim_{b\to 3} -2\sqrt{-b+3} + 2\sqrt{3} = 2\sqrt{3}$$

$$\lim_{a\to 3} \int_a^4 \frac{dx}{\sqrt{x-3}} = \lim_{a\to 3} \int_a^4 (x-3)^{1/2} = \lim_{a\to 3} \cancel{\frac{2(x-3)^{3/2}}{3}}\Big]_a^4$$

$$2\sqrt{x-3}\ \Big]_a^4$$

$$= \lim_{a\to 3} \left[\frac{2}{3}(4-3)^{3/2} - \frac{2}{3}(a-3)^{3/2}\right]$$

$$= \frac{2}{3} - 0 = \frac{2}{3}.$$

$$\Rightarrow \boxed{\int_0^4 \frac{dx}{\sqrt{|x-3|}} = 2\sqrt{3} + \frac{2}{3}}$$

$$\sum_{n=1}^{\infty} \frac{n!}{5^n}$$

n^{th} root test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$$\lim_{n\to\infty} \frac{(n+1)!}{5^{n+1}} \times \frac{5^n}{n!}$$

$$\frac{(n+1)}{5}$$

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$$

$$\ln\left(\frac{n+1}{n}\right)$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)$$

$$\frac{-\frac{1}{n^2}}{\frac{1}{1+\frac{1}{n}}} = -\frac{1}{n^2} \times \left(1+\frac{1}{n}\right)$$

$$= -\frac{1}{n^2} \cdot \left(\frac{n+1}{n}\right)$$

$$= \frac{\frac{-n-1}{n^3}}{-\frac{1}{n^2}} \cdot -n^2$$

$$\frac{n^3 \; n^2}{n^3}$$

20

8) ~~(20 points)~~ Determine whether the following series converges or diverges. Give a reason for your answer.

10

a) $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

using the n^{th} root test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a^n} = \lim_{n\to\infty} \frac{(n+1)!}{5^{n+1}} \times \frac{5^n}{n!} = \lim_{n\to\infty} \frac{(n+1)\cancel{n!}}{\cancel{5^{n+1}}} \times \frac{\cancel{5^n}}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{5} = \infty$$

$$\lim_{n\to\infty} \frac{a^{n+1}}{a^n} > 1$$

so this series diverges according to the ~~n^{th} root test~~ Ratio test,

10

b) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

we use the limit comparison test.

we ~~have~~ compare $\ln(1+\frac{1}{n})$ to $\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} \overset{l'hop}{=} \lim_{n\to\infty} \frac{-\frac{1}{n^2}}{\frac{1}{1+\frac{1}{n}}} \times (-n^2)$$

$$= \lim_{n\to\infty} \frac{-n-1}{n^3} \cdot (-n^2)$$

$$= \lim_{n\to\infty} \frac{n^3+n^2}{n^3} = 1$$

$$0 < 1 < \infty$$

and $\frac{1}{n}$ diverges because $p=1$

then we can say that $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n})$ diverges too.